On the Free Oscillation of Water in a Circular Lake with Convexed Paraboloidal Base

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1 Introduction

The problem of the free oscillation of water in a lake of uniform depth has been studied in great deal by many authors when it has a circular, rectangular or elliptic boundary. Meanwhile, Professor HIDAKA has given many not only important, but also interesting results by means of numerical integration in case of non-uniform depth.

When the depth is non-uniform, generally it is not so easy to find the exact solution of the fundamental equation which governs the phenomena. Thus the lake is mostly replaced by the one with constant depth to reduce the equation to the simpler one. This constant depth will be given as the mean value of the real depth with the area of the free surface of that lake. As the results of this replacement the boundary wall is to be vertical everywhere so in the free surface the water velocity along the normal at any point of the boundary must be zero. This is the boundary condition that should be imposed on the solution of the fundamental equation.

In the present paper, the fundamental normal mode of the free oscillation of water in a circular basin is exactly investigated when the form of the base is convexed paraboloid. The resulting relationship between the frequency of the fundamental normal mode of oscillation and the parameter which defines the shape of the base is shown graphically when the value of this parameter is not so different from one. The well-known value of the frequency for the circular basin with mean depth is also shown for the comparison. An approximate fromula for this relationship is also derived by using the same methed that Professor YAMADA has used to investigate the free oscillation of Lake Toya in Japan. It is one of the aim of this paper to estimate the accuracy of such method by comparing these approximate values of the frequency with those values obtained by the exact solution.

2 Geometric interpretaion

In the present case, we consider the circular basin whose base is convexed paraboloid so we may assume the depth h changes according to the law

$$h=h_0\{(2-\lambda)+2(\lambda-1)\frac{r^2}{a^2}\}, \tag{1}$$

where r is the distance from the cetner of the circular boundary in the free surface when it is undisturbed. In Fig. 1 AB shows undisturbed free surface of water and the thick line curve be the shape of the base in the cross section by the vertical plane through the center of the lake. The rectangle APQB shows the cross section of the circular basin with uniform depth whose volume is equal to that of the lake which we are now concerned with. Denote the coordinate of the point E $(a, -\lambda h_0)$ where a and h_0 being respectively the radius of the circular boundary and the mean depth, and λ a certain positive number between 1 and 2.

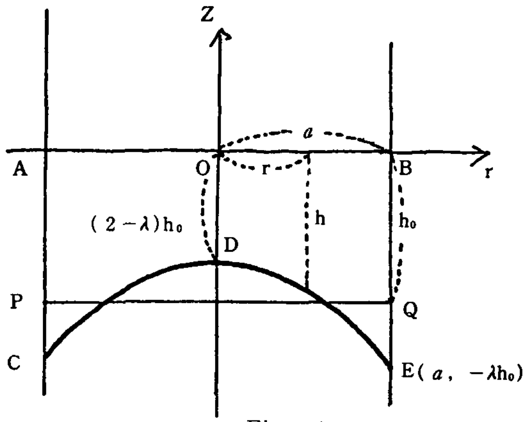

Fig. 1

Then from the relation (1) it is easily seen that the quantity of water contained in the surface generated by revolving the segment AC and the parabolic arc CD about z-axis is always equal to $\pi a^2 h_0$. Keeping both radius a and mean depth h_0 in constant we may obtain the various types of the paraboloidal bottom by varying the parameter λ from 1 to 2.

3 Fundamental equation and its solution

Now we shall proceed to get the fundamental equation and its solution. Let the rectangular axes (x, y) be taken in the free surface in undisturbed level in such a way that the origin coincides with the center of the boundary circle. Then if w be the vertical displacement of the free surface from its equilibrium position, the differential equation for determining the displacement of the free surface is given by

$$\frac{\partial^2 w}{\partial t^2}=g\,\mathrm{div}(h\ \mathrm{grad}\ w), \tag{2}$$

where g and h being respectively the acceleration due to gravity and the depth at any point on the bottom. Now h is supporsed to be a function of (x, y). When the free surface oscillates in a normal mode, the displacement w is of the form

$$w=\zeta(x,y)\cos\sigma t, \tag{3}$$

where ζ is a function of (x, y) and σ is a circular frequency. If we insert (3) into (2) we obtain

$$\mathrm{div}(h\ \mathrm{grad}\ \zeta)+\frac{\sigma^2}{g}\zeta=0. \tag{4}$$

In the present problem as h is a function of r only, the distance from the origin, introducing the polar coordinates (r, θ), (4) may be written

$$h\left(\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}\right)+\frac{dh}{dr}\frac{\partial\zeta}{\partial r}+\frac{\sigma^2}{g}\zeta=0. \tag{5}$$

Considering the normal modes of oscillation in which the motion is symmetrical about the center of the lake we obtain from both (5) and (1)

$$h_0\left\{(2-\lambda)+2(\lambda-1)\frac{r^2}{a^2}\right\}\left(\frac{d^2\zeta}{dr^2}+\frac{1}{r}\frac{d\zeta}{dr}\right)+4h_0(\lambda-1)\frac{r}{a^2}\frac{d\zeta}{dr}+\frac{\sigma^2}{g}\zeta=0. \tag{6}$$

Putting

$$b=\sqrt{\frac{2-\lambda}{2(\lambda-1)}}\,a, \tag{7}$$

that integral of the equation (6) which is finite at the origin is easily found in the following form of ascending series

$$\zeta=A\left\{1-\frac{p}{2^2}\frac{r^2}{b^2}+\frac{p(p+8)}{2^2\,4^2}\frac{r^4}{b^4}-\frac{p(p+8)(p+24)}{2^2\,4^2\,6^2}\frac{r^6}{b^6}\right.$$

$$\left.+\cdots\cdots\cdots\cdots+(-1)^n\frac{p(p+8)\cdots[p+(2n-2)(2n)]}{2^2\,4^2\cdots\cdots(2n)^2}\frac{r^{2n}}{b^{2n}}+\cdots\right\}, \tag{8}$$

where

$$\frac{\sigma a}{\sqrt{g h_0}}=\sqrt{2(\lambda-1)}\sqrt{p}, \tag{9}$$

and A being an arbitrary constant.

In the usual notaion of hypergeometric series the solution (8) can also be written as

$$\zeta=F\left(\alpha,\beta,\gamma,-\frac{r^2}{b^2}\right), \tag{10}$$

where

$$\alpha+\beta=1,\ \alpha\beta=\frac{p}{4},\ \gamma=1 \tag{11}$$

Since the boundary wall is vertical everywhere, as well known ζ must satisfy the following boundary condition

$$\left(\frac{\partial\zeta}{\partial r}\right)_{r=a}=0. \tag{12}$$

Substitution of (8) into (12) immediately gives the following relation

$$1-\frac{p+8}{8}\left(\frac{a}{b}\right)^2+\frac{(p+8)(p+24)}{8\cdot24}\left(\frac{a}{b}\right)^4-\frac{(p+8)(p+24)(p+48)}{8\cdot24\cdot48}\left(\frac{a}{b}\right)^6+\cdots\cdots=0. \tag{13}$$

For any circular basin whose depth varies according to the law (1), $\frac{a}{b}$ is fixed by (7) so by using the smallest positive root of the equation (13), the relation (9) will give the frequency of the fundamental normal mode of the free oscillation of this lake. The radius of the nodal circle is given as the smallest positive root of the equation $\zeta=0$ for the values of p determined above.

4 Numerical results

The writer has obtained the smallest positive root of the equation (13) in three cases in which the value of λ is equal to 1.1, 1.2 and 1.25, respectively. To this end the values of the left hand side of the equation (13) for different values of p have been calculated in each case and plotting these values against p the searching root has been determined graphically. The results are shown in Table 1. The radius R of the nodal circle in each case is given in row 4 of Table 1.

By using the relation (9) the curve of $\frac{\sigma a}{\sqrt{g h_0}}$ plotted against the parameter λ is shown by the thick line curve in Fig. 2, where is also shown the well-known value of $\frac{\sigma a}{\sqrt{g h_0}}$ equal to 3.832 for the circular basin with constant depth h_0.

Table 1. Values of λ, p, $\frac{\sigma a}{\sqrt{g h_0}}$ and R

λ	1	1.1	1.2	1.25
p		7.075	33.85	26.3
$\frac{\sigma a}{\sqrt{g h_0}}$	3.832	3.762	3.680	3.63
R	0.628@	0.621@	0.615@	0.612@

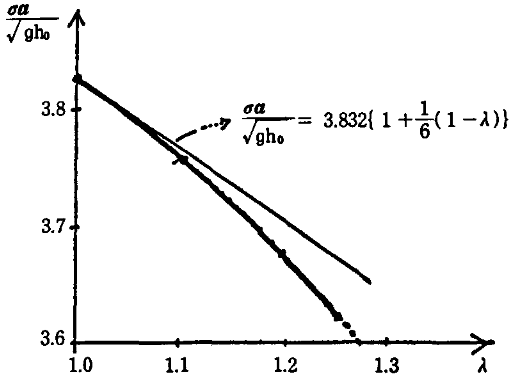

Fig. 2

As easily seen when λ approaches $\frac{4}{3}$, $\frac{a}{b}$ tends to 1 by (7) so the convergency of the series in the lefthand side of (13) gets slow. Thus in these cases it seems very laborious to find the root of the equation (13) without the aid of computer.

5 The approximate value of the frequency

Now we shall proceed to explain the method to compute the approximate value of the frequency when the depth is almost uniform. This method had often been used in wave mechanics and was introduced by Dr. H. YAMADA into the study of the free oscillation of Lake Toya.

Denoting the mean depth of the lake h_0, we may express the depth h at any point (x, y) in the following form

$$h=h_0(1+\varepsilon), \tag{14}$$

with

$$\int_s\int \varepsilon dxdy=0, \tag{15}$$

where $|\varepsilon|$ is a small quantity compared with unity, and s denotes the area of the free surface of the lake. Substituting (14) into (4) we obtain

$$\nabla^2\zeta+\mathrm{div}(\varepsilon\,\mathrm{grad}\zeta)+\mu\zeta=0, \tag{16}$$

where

$$\mu=\sigma^2/g h_0. \tag{17}$$

If the boundary wall of the lake is vertical everywhere, as well-hnown ζ should satisfy the following boundary condition

$$\left(\frac{\partial\zeta}{\partial n}\right)_{\text{on the boundary}}=0, \tag{18}$$

where $\frac{\partial}{\partial n}$ indicates the differentiation along the normal to the boundary curve. In case $|\varepsilon|$ is small enough we may assume ζ and μ are respectively expressed as

$$\begin{cases} \zeta=\zeta_0+\zeta_1, & (19) \\ \mu=\mu_0+\mu_1, & (20) \end{cases}$$

where ζ_0 and μ_0 being those of ζ and μ when the depth is uniform ($\varepsilon=0$). Meanwhile ζ_1 and μ_1 may be understood as the small correction terms due to non-uniformity of the depth, which are supposed to be in the same order as ε. Inserting (19) and (20) into (16) and equating the terms whose order are ε^0 and ε to zero respectively, we have

$$\begin{cases} \nabla^2\zeta_0+\mu_0\zeta_0=0, & (21) \\ \nabla^2\zeta_1+\mu_0\zeta_1=-\mu_1\zeta_0-\mathrm{div}(\varepsilon\,\mathrm{grad}\zeta_0). & (22) \end{cases}$$

The equation (21) is obviously the one in case of uniform depth. We impose the boundary condition (18) to both ζ_0 and ζ_1 as in the following

$$\begin{cases} \left(\frac{\partial\zeta_0}{\partial n}\right)_{\text{on the boundary}}=0, & (23) \\ \left(\frac{\partial\zeta_1}{\partial n}\right)_{\text{on the boundary}}=0. & (24) \end{cases}$$

By using Green's formula in two dimensions for two functions ζ_0 and ζ_1 we have

$$\int_s\int(\zeta_0\nabla^2\zeta_1-\zeta_1\nabla^2\zeta_0)\mathrm{dxdy}=\oint_\ell\left(\zeta_0\frac{\partial\zeta_1}{\partial n}-\zeta_1\frac{\partial\zeta_0}{\partial n}\right)d\ell, \tag{25}$$

when ℓ denotes the length along the boundary curve. According to the boundary conditions (23) and (24) the right hand side of (25) is equal to zero. Then inserting the relation (21) into the integrand on the left hand side of (25) we obtain

$$\int_s\int\zeta_0(\nabla^2\zeta_1+\mu_0\zeta_1)\mathrm{ds}=0, \tag{26}$$

Remembering the relation (22), after a simple calculation we obtain ultimately the following expression for μ_1

$$\mu_1=-\frac{\int_s\int\zeta_0\mathrm{div}\,(\varepsilon\,\mathrm{grad}\zeta_0)\mathrm{ds}}{\int_s\int\zeta_0^2\mathrm{ds}}=\frac{\int\int\varepsilon(\mathrm{grad}\zeta_0)^2\mathrm{ds}}{\int\int\zeta_0^2\mathrm{ds}}. \tag{27}$$

Now we shall apply this method to the present problem. From the relation (1) we have

$$h=h_0\left\{1+(1-\lambda)\left(1-\frac{2r^2}{a^2}\right)\right\}. \tag{28}$$

In case when the base is slightly different from flat plane the parameter λ is nearly equal to 1 so we may put

$$\varepsilon=(1-\lambda)\left(1-\frac{2r^2}{a^2}\right), \tag{29}$$

For the case which we are now concerned with, it is well known that ζ_0 and μ_0 will be given by the following formulae (30) and (31), respectively,

$$\zeta_0 = J_0(\sqrt{\mu_0}\, r) = J_0\left(\frac{\alpha_1}{a} r\right), \tag{30}$$

$$\mu_0 = \frac{\alpha_1^2}{a^2}, \tag{31}$$

where $\alpha_1 = 3.832$ is the smallest positive root of the equation

$$J_1(\xi) = 0. \tag{32}$$

$J_0(\xi)$ and $J_1(\xi)$ are Bessel functions of the zeroth and the first order, respectively. Substituting (29) and (30) into (27), and putting

$$\rho = \frac{r}{a},$$

we get

$$\mu_1 = (1-\lambda)\left\{\frac{\int_0^1 (\rho - 2\rho^3)[J_1(\alpha_1\rho)]^2 d\rho}{\int_0^1 \rho [J_0(\alpha_1\rho)]^2 d\rho} \frac{\alpha_1^2}{a^2}\right\}. \tag{33}$$

Meanwhile following results can easily be found,

$$\int \xi [J_0(\xi)]^2 d\xi = \frac{1}{2}\xi^2\{[J_0(\xi)]^2 + [J_1(\xi)]^2\}, \tag{34}$$

$$\int \xi [J_1(\xi)]^2 d\xi = \frac{1}{2}\xi^2\{[J_0(\xi)]^2 + [J_1(\xi)]^2\} - \xi J_0(\xi) J_1(\xi), \tag{35}$$

$$\int \xi^3 [J_1(\xi)]^2 d\xi = \frac{1}{6}\xi^4\{[J_0(\xi)]^2 + [J_1(\xi)]^2\} - \frac{2}{3}\xi^3 J_0(\xi) J_1(\xi) + \frac{2}{3}\xi^2 [J_1(\xi)]^2 \tag{36}$$

Thus using these relations and considering that α_1 is the root of the equation

$$J_1(\xi) = 0, \tag{32}$$

we have the following expressiin for μ_1

$$\mu_1 = (1-\lambda)\frac{\frac{1}{2}[J_0(\alpha_1)]^2 - \frac{1}{3}[J_0(\alpha_1)]^2}{\frac{1}{2}[J_0(\alpha_1)]^2}\frac{\alpha_1^2}{a^2} = \frac{1}{3}(1-\lambda)\frac{\alpha_1^2}{a^2}. \tag{37}$$

Inserting (31) and (37) into (20) and considering the relation (17) we have finally

$$\frac{\sigma a}{\sqrt{g h_0}} = \alpha_1\left\{1 + \frac{1}{3}(1-\lambda)\right\}^{\frac{1}{2}} \doteqdot \alpha_1\left\{1 + \frac{1}{6}(1-\lambda)\right\} = 3.832\left\{1 + \frac{1}{6}(1-\lambda)\right\}. \tag{38}$$

The values of $\frac{\sigma a}{\sqrt{g h_0}}$ calculated by this approximate formula are shown by a thin line in Fig. 2. It will be seen that in our case, this approximate formula (38) is particularly useful for calculating the numerical value of $\frac{\sigma a}{\sqrt{g h_0}}$ when λ does not so differ from. 1.

6 Summary

The fundamental normal mode of the free oscillation of water in a circular lake with convexed paraboloidal base is exactly investigated in case when the vertical displacement of the water surface from the equilibrium position is to be symmetrical about the center of the free surface.

The relationship between $\frac{\sigma a}{\sqrt{g h_0}}$ and λ is shown graphically for the values of λ which are not so different from 1, when σ is a circular frequency, a a radius of the circular boundary, λ a parameter that determines the shape of the base and g and h_0 being the acceleration due to gravity and the mean depth of the lake respectively. When λ takes a value between 4/3 and 2, it is unable to find the value of $\frac{\sigma a}{\sqrt{g h_0}}$ because the series in (13) does not converge. In such a case some other methods of analysis should be searched for.

An approximate formula between $\frac{\sigma a}{\sqrt{g h_0}}$ and λ is also derived, and besides, the radii of the nodal circles for three cases have been determined.

It has been well illustrated that the parameter λ has little influence not only on the frequency but also on the radius of the nodal circle when its value does not so differ from 1. It is also shown that YAMADA's method to find the approximate value of the frequency of the free oscillation of the lake with nearly uniform depth is particularly useful in case when the shape of the base is convexed paraboloid.

References

Hidaka, K. (1931): The free oscillations of water in spindle shaped and elliptic basins as well as the associated problems. (Problem of water oscillations in various types of basins and canals. Part I) Mem. Imp. Mar. obs. (Kobe, Japan), **4**, 99-219.

Hidaka, K. (1936): On the theoretical investigation of the free oscillation of water in Lake Toya. Umi to Sora, **16**, 376-406 (in Japanese)

Hidaka, K. (1931): On the free oscillation of water in the Osaka Bay. Umi to Sora, **17**, 271-283, (in Japanese)

Yamada, H and Adachi, A (1936): Seiches in Lake Toya. Umi to Sora, **16**, 29-38. (in Japanese)