On a Sum of one Double Series

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§ 1 Introduction

In some cases when we try to get a solution of a certain physical problem, we happen to find two solutions of different types, according to the methods of analysis. It seems true that such cases hardly occur, but on such occasions each of two formulae expressing the solution gives the answer for the same problem, therefore if the appearance of the fomulae may entirely differ, the physical quantities given by them should have the same value. Accordingly, one of the formulae must be absolutely identical with the other. Thus, we are able to find one relation.

In the present paper one of such examples is shown for the conduction of heat in a cube of homogeneous isotropic substance.

§ 2 The steady flow of heat in a cube of homogeneous isotropic substance.

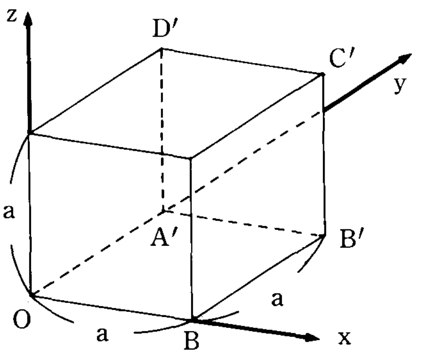

Fig. 1

In Fig.1 the cube is bounded by the planes

$$x = 0, \quad x = a;$$
$$y = 0, \quad y = a;$$
$$z = 0, \quad z = a.$$

In steady state the equation for the temperature U is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \qquad (1)$$

We take the following surface conditions

$$(U)_{z=0} = f(x,y),$$

and the other faces at zero, where $f(x,y)$ is a given function of x and y.

The solution in this case is easily obtained as

$$\left.\begin{aligned} U&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\,\sinh\frac{\sqrt{m^2+n^2}}{a}\pi(a-z)\sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{a}y,\\ \text{where}\quad A_{mn}&=\frac{4}{a^2\sinh\sqrt{m^2+n^2}\pi}\int_0^a\int_0^a f(\lambda,\mu)\,\sin\frac{m\pi}{a}\lambda\,\sin\frac{n\pi}{a}\mu d\lambda d\mu. \end{aligned}\right\} \tag{2}$$

By the use of above formulae (2), when the face O B B' A' is always kept at constant temperature T_0 and all other faces at zero, putting $f(x,y)=T_0$, the temperature U_M at the center of the cube is given in the following.

$$U_M=\frac{8T_0}{\pi^2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}}{(2r+1)(2s+1)\cosh\frac{1}{2}\sqrt{(2r+1)^2+(2s+1)^2}\pi}. \tag{3}$$

Now we maintain the temperature of any surface of the cube at constant and all the others at zero. As a cube has six faces we have six such cases.

Let express the solution in each case by U_1, U_2, $\cdots$ and U_6, respectively. Now according to the geometrical interpretaion it is clear that any of these solutions gives the same value for the temperature at the center of the cube.

By the way the equation (1) is linear so the formula (4) written below must be the solution in the case all faces of a cube are kept at constant temperature T_0,

$$U = U_1 + U_2 + \cdots\cdots + U_6. \tag{4}$$

By the reasoning mentioned above the temperature $\overline{U}_M$ at the center of the cube can be written as

$$\overline{U}_M=6U_M=6\times\frac{8T_0}{\pi^2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}}{(2r+1)(2s+1)\cosh\frac{1}{2}\sqrt{(2r+1)^2+(2s+1)^2}\pi}. \tag{5}$$

On the other hand, in the case we are now concerned with the temperature $\overline{U}_M$ can be found at once. Evidently the temperature at any point in the cube is T_0, so the temperature $\overline{U}_M$ can also be written down as

$$\overline{U}_M = T_0. \tag{6}$$

It follows from (5) and (6) that

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r+s}}{(2r+1)(2s+1)\cosh\frac{1}{2}\sqrt{(2r+1)^2+(2s+1)^2}\pi}=\frac{\pi^2}{48}. \tag{7}$$

Thus we are now able to obtain the sum of the double series defined on the left-hand side of the above formula (7).

However, it seems rather hard to derive the relation (7) directly.

§3 Summary

If we could find the two different expressions for the solution of a certain physical problem, by identifying one of them with the other we would find one mathematical relation.

In the present paper, concerning the problem of steady flow of heat in a homogeneous isotropic cube one such example is introduced.

Appendix

The similar problem in two dimensions is to find the temperature U_M at the center of homogeneous isotropic thin square plate when the temperature of its three sides are all kept at zero and the fourth one at constant T_0.

Using the solution of Laplace's equation in two dimensions which satisfies the boundary conditions mentioned above, we get the result at once in the following

$$U_M=\frac{2T_0}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)\cosh\frac{2r+1}{2}\pi}. \tag{8}$$

On the other hand by the similar reasoning as in the case of a cubic solid, we have the relation

$$U_M = \frac{1}{4}T_0. \tag{9}$$

Thus we get the following formula

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)\cosh\frac{2r+1}{2}\pi}=\frac{\pi}{8}. \tag{10}$$

By the way we can derive this formula directly, unlike the three dimensional case, by means of a contour integral and residue theorem of Cauchy.

We take the function $f(z)$ of a complex variable z as

$$f(z)=\frac{1}{z\cosh\frac{\pi}{2}z\cos\frac{\pi}{2}z}, \quad z=x+iy.$$

By using a contour integral of above $f(z)$ over the square path along the lines

$$x=\pm 2n, \quad y=\pm 2n, \qquad \text{(Fig. 2)}$$

where n is a certain positive integer, and letting n tend to infinity we obtain (10). On carrying out above calculation along any side of the square path the following inequality is available,

$$|f(z)| \leqq \frac{4}{e^{n\pi} - e^{-n\pi}}.$$

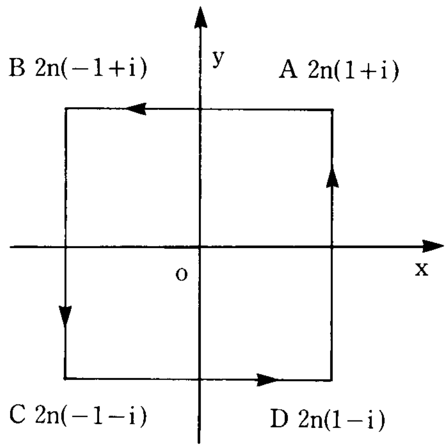

Fig. 2